straight muscle by similar regions of the cortex of the *same* side as the muscle. A point of difference—and it is a suggestive one—between the two cases appears to be that under inhibitory relaxation of the *rectus externus* from cortical excitation the globus may rotate beyond the middle line of the palpebral fissure, whereas under cortical relaxation of the *rectus internus* the eyeball may travel up to that middle line, but very rarely, if ever, trespasses beyond it.

It thus seems clear that by experiment abundant support can be obtained for the supposition put forward by Charles Bell.

V. "On the Differential Covariants of Plane Curves, and the Operators employed in their Development." By R. F. Gwyther, M.A., Fielden Lecturer in Mathematics, Owens College, Manchester. Communicated by Professor Horace Lamb, F.R.S. Received April 14, 1893.

(Abstract.)

Consider any point (x, y) on a standard plane curve, and write a_1, a_2, a_3, &c., for dy/dx, $d^2y/dx^2 . 2!$, $d^3y/dx^3 . 3!$, &c., the differential coefficients being derived from the equation to the curve. Let $(\xi . \eta)$ be the current coordinates of a point which moves so that $f(\xi, \eta, x, y, a_1, a_2, a_3, \ldots) = 0$, say, on a trajectory of the standard curve. Now let a general homographic transformation affect ξ, η, and x, y alike; the function which replaces $f(\xi, \eta, x, y, a_1, \ldots)$ will generally entirely change its character; if, however, it retains the same form (except that it is affected by a certain factor of which the form is to be found), I define it to be a differential covariant of the standard curve.

It is obvious that among the covariants will be found tangents and polar curves, and the ordinary covariant curves. It is proposed to investigate the subject generally, and to find the relations of the covariant with the contravariant curves.

§ 1.

For the purpose of obtaining the forms of the linear partial differential equations which express the conditions that a function should be a covariant function, an infinitesimal homographic transformation is employed, expressed by the relations

$$\frac{x}{X+B_1Y} = \frac{y}{Y} = \frac{1}{AX+BY+1},$$

where A, B, and B_1 are small, with identical relations for ξ and η.

The conditions found may be stated as follows.

Simple Conditions.

Call the algebraic degree of the equation in ξ η, d, and the algebraic degree of a coefficient in $a_1, a_2, a_3, \ldots, d_x$, which we shall call the degree of the coefficient. In the same coefficient call the sum of the indices of differentiation the weight of the coefficient, and write it w. Write D for the determinant of the homographic substitution,

$$\lambda = (AX+BY+1)(1+B_1Y_1)-(X+B_1Y)(A+BY_1),$$

$$\mu = AX+BY+1,$$

$$\nu = A\xi+B\eta+1.$$

Then (1) the form of the multiplying factor is

$$D^{d+d_x} \,.\, \lambda^{-(d+d_x+w)} \,.\, \mu^{2w-d_x} \,.\, \nu^{-d};$$

(2) ξ, η, x, y, and a_1 only enter in the forms $\xi-x$ and $\eta-y-a_1(\xi-x)$, which will be written $\pi-p$;

(3) The function is homogeneous in $\eta-y$ and the several differential coefficients of y;

(4) The weight of the coefficient of each power of $\xi-x$ is uniform, *i.e.*, each coefficient is isobaric, and this weight, diminished by the index of the power of $\xi-x$, is uniform throughout the function;

Conditions in the form of Linear Partial Differential Equations.

(5) The other equations of condition take the forms

$$(\pi-p)\frac{\partial f}{\partial(\xi-x)} = SSpa_pa_{n-p+1}\frac{\partial f}{\partial a_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (A)$$

$$(\xi-x)\left\{(\xi-x)\frac{\partial f}{\partial(\xi-x)}+(\pi-p)\frac{\partial f}{\partial(\pi-p)}-df\right\} = S(n-2)\,a_{n-1}\frac{\partial f}{\partial a_n} \quad \ldots\ldots\ldots \quad (B),$$

$$(\pi-p)\left\{(\xi-x)\frac{\partial f}{\partial(\xi-x)}+(\pi-p)\frac{\partial f}{\partial(\pi-p)}-df\right\}$$

$$= SS\,(p-1)\,a_pa_{n-p}\frac{\partial f}{\partial a_n} \ldots\ldots\ldots \quad (C),$$

where S denotes summation for all values of n which introduce values of a_n from a_2 upwards, and SS denotes a similar double summation, first with regard to p, which introduces values of a_p between a_2 and a_n, and secondly with regard to n, which introduces values of a_n from a_2 upwards.

These equations will be abbreviated to

$$O_1 f = \Omega_1 f \ \ldots\ldots\ldots\ldots\ldots\ldots \ \text{(A)}.$$

$$O_2 f = \Omega_2 f \ \ldots\ldots\ldots\ldots\ldots\ldots \ \text{(B)}.$$

$$O_3 f = \Omega_3 f \ \ldots\ldots\ldots\ldots\ldots\ldots \ \text{(C)}.$$

§ 2.—*Interpretation of the Conditions* (A), (B), *and* (C).

Write ${}_r\phi_q\,(\pi-p)^r\,(\xi-x)^q$ as a type of the terms in the covariant.

Then (1) $O_1 f = \Omega_1 f$ becomes

$${}_r\phi_q = \Omega_1\,{}_{r+1}\phi_{q-1}/q,$$

so that, in any set of terms homogeneous in $(\pi-p)$ and $(\xi-x)$ the coefficients may successively be derived from that of the term in $(\pi-p)^{r+q}$ by means of the operator Ω_1.

And (2) $O_2 f = \Omega_2 f$ becomes

$${}_r\phi_q = -\Omega_2\,{}_r\phi_{q+1}/d-r-q,$$

so that the coefficient of such term homogeneous of degree $r+q$ in $(\pi-p)$ and $(\xi-x)$ may be derived from the coefficients of the set of terms homogeneous of degree $r+q+1$ in $(\pi-p)$ and $(\xi-x)$.

Also (3) $O_3 f = \Omega_3 f$ becomes

$${}_r\phi_q = -\Omega_3\,{}_{r+1}\phi_q/d-r-q.$$

We thus have a second mode of derivation of character similar to the last, showing that the operators are not all independent.

The order of derivation is shown by the chart

$$\begin{tikzcd}
{}_d\phi_0 \arrow[rr] \arrow[dr] & & {}_d\phi_1 \arrow[rr] \arrow[dl] \arrow[dr] & & {}_d\phi_2 \arrow[rr] \arrow[dl] \arrow[dr] & & {} \\
& {}_{d-1}\phi_0 \arrow[rr] \arrow[dr] & & {}_{d-1}\phi_1 \arrow[rr] \arrow[dl] \arrow[dr] & & {}_{d-1}\phi_2 \arrow[rr] \arrow[dl] & & {} \\
& & {}_{d-2}\phi_0 \arrow[rr] & & {}_{d-2}\phi_1 \arrow[rr] & & {}
\end{tikzcd},$$

where $\longrightarrow$ denotes deduction by the operator Ω_1,

$\swarrow$,, ,, Ω_2,

$\searrow$,, ,, Ω_3,

in which it is easy to trace the relations

$$\Omega_2\Omega_1 - \Omega_1\Omega_2 = \Omega_3,$$

$$\Omega_2\Omega_3 - \Omega_3\Omega_2 = 0,$$

$$\Omega_3\Omega_1 - \Omega_1\Omega_3 = 0.$$

From this it is clear that ${}_d\phi_0$, the coefficient of the highest power of $(\pi-p)$, satisfies the differential equation $\Omega_2 f = 0$, and that the whole expansion can be found if ${}_d\phi_0$ is known.

On this account I call ${}_d\phi_0$ the source or matrix of the covariant, and investigate the solutions of $\Omega_2 f = 0$.

It is next proved that

$$\Omega_2 \frac{df}{dx} - \frac{d}{dx}\,\Omega_2 f = (2w - d_x) f,$$

and hence that, if f is a homogeneous, isobaric function of $a_2, a_3, \ldots,$ such that $\Omega_2 f = 0$, and that $2w - d_x = 0$, then df/dx is equally a solution of $\Omega_2 f = 0$.

If we know two solutions of $\Omega_2 f = 0$, solutions of all higher orders can then be educed by the process here implied.

$\Omega_2 = 0$ on expansion becomes

$$a_2 \frac{\partial}{\partial a_3} + 2a_3 \frac{\partial}{\partial a_4} + 3a_4 \frac{\partial}{\partial a_5} + \ldots\ldots = 0,$$

and therefore we have two solutions readily, viz.,

$$u_2 = a_2,$$

$$u_4 = a_2 a_4 - a_3^2.$$

The educts successively found by this method are, however, not generally irreducible, and it is shown how to find the irreducible solutions for the successive orders, which are written u_2, u_4, u_5, &c. Any common solution of $\Omega_2 f = 0$ and $\Omega_1 f = 0$ is a differential invariant, and not the matrix of a covariant. For the second and fifth orders, the seventh and all orders higher than the seventh, there is a differential invariant, and for the sixth there is a common solution of $\Omega_2 f = 0$ and $\Omega_3 f = 0$, which I write L_6. It is the matrix of a straight line through x, y, and all matrices may be expressed as functions of u_4, L_6, and differential invariants. The order of the covariant can be inferred from the mode in which u_4 and L_6 enter the matrix.

§ 3.

If the systems of coefficients are subjected to a reciprocal transformation, of which $\alpha\xi - \pi - \gamma = 0$ may be taken as the typical relation, there is this relation between the operators Ω_1 and Ω_2—if u is a solution of either $\Omega_1 f = 0$ or $\Omega_2 f = 0$ in which $a_2, a_3, \ldots,$ are the arguments, and if, in consequence of the substitution for these qualities of $A_2, A_3, \ldots.$ (similar functions of the correlative system of coordinates), u becomes U, then U is a solution of the other corresponding equation, that is, of $\Omega_2 F = 0$ or $\Omega_1 F = 0$ respectively.

Or, briefly, we may say the result is the interchange of the operators Ω_1 and Ω_2. Regarded as functions of A_2, A_3, &c., the coefficients in a contravariant are developed from a matrix satisfying $\Omega_2 F = 0$, by the process which has been already described. Regarded as a function of a_2, a_3, &c., they are derived from a matrix satisfying $\Omega_1 f = 0$, by a process which differs from that previously described by the interchange of the operators Ω_1 and Ω_2. Any function of the coefficients in a covariant function which would be an invariant for a change of coordinates, but not for a homographic transformation, will be the matrix of a contravariant function. The matrix of the reciprocal is the discriminant of the highest group of homogeneous terms in π and ξ treated as a binary quantic.

§ 4.

To apply these results to the cubic osculating the standard curve at the point (x, y), remove the origin temporarily to that point. The condition of intersecting the standard curve in a number of points coincident with this temporary origin is an invariantal relation, and we may treat the coefficients in the equation as if they contained differential invariants only.

The form of the matrix of the cubic is

$$fu_4^2 + (\phi + \phi_1 L_6 + \phi_2 L_6^2)\, u_4 + \psi + \psi_1 L_6 + \psi_2 L_6^2 + \psi_3 L_6^3 + \psi_4 L_6^4,$$

with the condition $\qquad f + u_5^2\phi_2 + u_5^4\psi_4 = 0,$

where all the functions contain differential invariants only, and, retaining only invariantal coefficients, the expanded equation is

$$\psi\pi^3 - u_2^2 u_5 \psi_1 \pi^2\xi + u_2^4 (\phi + u_5^2\psi_2)\,\pi\xi^2 - u_2^6 u_5 (\phi_1 + u_5^2\psi_3)\,\xi^3$$
$$- u_2^3\phi\pi^2 + u_2^2 u_5 \phi_1 \pi\xi - u_2^7 (2f + u_5^2\phi_2)\,\xi^2 + u_2^6 f\pi = 0,$$

and writing the differential invariants of the several orders u_2, u_5, U_7, U_8, U_9, we have for the shape of the standard curve near the origin

$$\pi = u_2\xi^2 + \frac{u_5}{u_2^2}\xi^5 + \frac{U_7}{u_2^4 u_5}\xi^7 + \frac{U_8 + 10u_5^4}{u_2^6 u_5^2}\xi^8 + \frac{u_5^2 U_9 - 3U_7^2}{u_2^6 u_5^3}\xi^9 + \ldots .$$

Substituting in the equation of the cubic, we find

$$\psi_2 = \psi_3 = \psi_4 = 0 \qquad \psi_1 = f, \qquad \psi = -u_5^2\phi_1, \qquad f = -u_5^2\phi_2,$$

and $\qquad u_5^2\phi = U_7 f, \qquad U_7\psi = V_8 f,$

where V_8 stands for $U_8 + 8u_5^4$.

If the cubic is non-singular this determines all the functions in the matrix, and gives the differential equation

$$u_5^2 U_7 U_9 - V_8^2 - 4U_7^3 + u_5^4 V_8 = 0.$$

The matrix of the equation to the tangents to the cubic from the origin is

$$(\phi + \phi_1 L_6 + \phi_2 L_6^2)^2 - 4f(\psi + \psi_1 L_6),$$

and the condition that the cubic may be nodal or cuspidal is that this matrix, as a function of L_6, may have a linear factor twice or thrice repeated.

In the case of the nodal cubic, the differential equation will be derived from $U_7 \psi = V_8 f$, and, putting $\psi = 2fk$, k is a root of

$$U_7 k^4 - \tfrac{1}{2} u_5^4 k^3 + 2U_7^2 k^2 - \tfrac{9}{2} u_5^4 U_7 k + U_7^3 + \tfrac{27}{16} u_5^8 = 0,$$

and from this the differential equation is found.

In the case of the cuspidal cubic, put $\psi = 2fk$, $u_5^2 \phi = fq$.

Then
$$16 k^3 = 27 u_5^4,$$
$$256 q^3 = 27 u_5^8,$$

and the differential equation is

$$256 U_7^3 - 27 u_5^8 = 0.$$

§ 5.

In this section it is attempted to develop a geometrical method, founded on the covariant theory.

The general equation to a covariant line takes the form

$$(u_5^2 \phi_3 u_4 + \phi_1 + \phi_2 L_6 - \phi_3 L_6^2)\, \pi - u_2^2 u_5 \{(\phi_2 - 2\phi_3 L_6) - u_5 \phi_3 a_3\}\, \xi - u_2^3 u_5^2 \phi_3 = 0,$$

and depends upon the invariant ratios $\phi_1 : \phi_2 : \phi_3$.

If we take a second covariant line, the coordinates of the point of intersection take the form

$$\pi = u_2^3 A / u_4 A + C + B a_3,$$
$$\xi = u_2 B / u_4 A + C + B a_3,$$

where

$$A = -u_5^2 (\phi_2 \phi'_3 - \phi'_2 \phi_3),$$
$$B = u_5 \{(\phi_3 \phi'_1 - \phi'_3 \phi_1) - (\phi_2 \phi'_3 - \phi'_2 \phi_3) L_6\},$$
$$C = (\phi_1 \phi'_2 - \phi'_1 \phi_2) + 2(\phi_3 \phi'_1 - \phi'_3 \phi_1) L_6 - (\phi_2 \phi'_3 - \phi'_2 \phi_3) L_6^2.$$

Or, more shortly,

$$A = u_5^2 \lambda,$$
$$B = u_5 (\mu - \lambda L_6),$$
$$C = \nu + 2\mu L_6 - \lambda L_6^2,$$

and its position depends upon the invariant ratios $\lambda : \mu : \nu$. Treating $(\lambda : \mu : \nu)$ as determining the position of a point, and $(\phi_1 : \phi_2 : \phi_3)$ the position of a line, the condition that $(\lambda : \mu : \nu)$ may lie on $(\phi_1 : \phi_2 : \phi_3)$ is

$$\lambda\phi_1 + \mu\phi_2 + \nu\phi_3 = 0.$$

Hence they form a correlative system of point and line coordinates. I define $(\lambda : \mu : \nu)$ as the invariantal coordinates of a point of homographic persistence, and $(\phi_1 : \phi_2 : \phi_3)$ as the invariantal coordinates of a covariant line.

The condition that $(\lambda : \mu : \nu)$ may lie upon a covariant curve of the n^{th} order will be an invariantal relation, homogeneous of the n^{th} degree in λ, μ, ν, between λ, μ, ν and the invariants in the coefficients of the equation to the curve (we may say the invariantal coordinates) of the curve.

If $f(\lambda : \mu : \nu) = 0$ expresses this relation, it is in this system of coordinates the equation to the curve; I call it the intrinsic invariantal equation to the curve.

The coordinates of the tangent to the curve at $(\lambda : \mu : \nu)$ are $\left(\frac{\partial f}{\partial \lambda} : \frac{\partial f}{\partial \mu} : \frac{\partial f}{\partial \nu}\right)$ and $\left(\lambda' \frac{\partial}{\partial \lambda} + \mu' \frac{\partial}{\partial \mu} + \nu' \frac{\partial}{\partial \nu}\right) f = 0$ is the equation to the first polar of $(\lambda' : \mu' : \nu')$ with respect to the curve.

Also writing x for μ/ν and y for λ/ν, the relations between π, ξ and x, y are of the form of a homographic transformation, and therefore any function of $d^2\pi/d\xi^2$, &c., which is a differential invariant, is equal to the identical function of d^2y/dx^2, &c., affected by a factor of known form. Hence, treating $f(\lambda : \mu : \nu) = 0$ or $f(x, y) = 0$ as an ordinary algebraic equation, it will have the same homographic singularities as the original covariant function, while the coefficients are the differential invariants which characterise the curve.

The intrinsic invariant equation to the osculating conic is

$$\lambda\nu + \mu^2 = 0, \qquad \text{or} \qquad y + x^2 = 0,$$

and to the non-singular osculating cubic is

$$u_5^4\lambda^2(V_8\lambda + U_7\mu) + (U_7^2\lambda - V_8\mu + U_7\nu)(\lambda\nu + \mu^2) = 0.$$

To terminate the abstract, the equation to the polar conic of the origin is

$$U_7^2\lambda^2 - V_8\lambda\mu + U_7\mu^2 + 2U_7\lambda\nu = 0,$$

and therefore $U_7^2\lambda - V_8\mu + U_7\nu = 0$ is the equation to the common chord of this conic and the osculating conic, and it touches the cubic at $\lambda = 0$, $\mu : \nu = U_7 : V_8$, the tangential of the origin. Also the second tangential of the origin lies on

$$V_8\lambda + U_7\mu = 0,$$

i.e., on the common chord of the cubic and the conic of closest contact.

The third tangential of the origin has the coordinates

$$U_7^2 : -u_5^4 U_7 : -(U_7+u_5^8),$$

which are independent of V_8. As is known, this is the corresidual of the eight consecutive points at the origin on all the several cubics for which V_8 is arbitrary.

The line $V_8\mu - U_7\nu = 0$, on which the tangential of the origin lies, passes through $(U_7^2 : -U_7V_8 : -V_8^2)$, the sixth point in which the osculating conic meets the cubic again.

The Society adjourned over Ascension Day to Thursday, May 18.

Presents, May 4, 1893.

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